Orthogonal Martingales

1. Let $\mathcal{M}^2$ be the set of L^2 martingales adapted to the filtration $(\mathcal{F}_t)$ on the stochastic base,

$$(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t), [0,\infty)), \text{"usual conditions"}$$

and such that $\sup_t \|X_t\|_2 < \infty$ for $X \in \mathcal{M}^2$. This means $X_t = M_t(X_\infty)$ for some X_∞. By $\mathcal{M}^2_0$ we mean the subset of $\mathcal{M}^2$ comprising martingales, (X_t), for which $X_0 = 0$.

There is a one to one correspondance between elements of $\mathcal{M}^2$ and elements, U, of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ given by,

$$L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \ni U \longmapsto (M_t(U)) \in \mathcal{M}^2.$$

This correspondance, restricted to $\mathcal{M}^2_0$ identifies $\mathcal{M}^2_0$ with the closed subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ comprising random variables with zero expectation.

In $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ there is a natural notion of orthogonality furnished by the inner product; for $U, V \in L^2$,

$$\langle U, V \rangle \equiv \int_\Omega UV \, d\mathbb{P} = 0.$$

Since each of U and V correspond to martingales in $\mathcal{M}^2$ it is natural to ask if this idea of orthogonality might be transfered to $\mathcal{M}^2$ and become a useful tool? It turns out that this idea has a limited use. However, there is a notion of orthogonality for elements of $\mathcal{M}^2$, which is "far stronger" than the one discussed above, and this is a useful tool in Stochastic Calculus. Here it is,

1.1 **Definition.** Let $(X_t), (Y_t)$ belong to $\mathcal{M}_0^2$. We say (X_t) and (Y_t) are (strongly) orthogonal if their 'product' is in $\mathcal{M}_0^1$, that is,

$(X_t Y_t)$ is an L^1-martingale adapted to $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t), [0,T])$, and $X_0 Y_0 = 0$.

We write, $(X_t) \perp (Y_t)$, for this relationship.

Let X be an L^2-martingale and τ a stopping time. Suppose τ takes the values $0 \leq t_0 < t_1 < \dots < t_n \leq \infty$, so $\tau = \sum_{i=0}^{n} t_i I_{E_i}$, $E_i = \{\tau = t_i\} \in \mathcal{F}_{t_i}$.

$X_\tau = \sum_{i=0}^{n} X_{t_i} I_{E_i}$ and if $t_j \leq t < t_{j+1}$ then

$$X_{\tau\wedge t} = \sum_{i=0}^{j} X_{t_i} I_{E_i} + X_t I_{\{\tau \geq t_{j+1}\}}.$$ Now let

$t_k \leq s < t_{k+1}$ with $s < t$. Consider $M_s(X_{\tau\wedge t})$.

$$M_s(X_{\tau\wedge t}) = \sum_{i=0}^{k} X_{t_i} I_{E_i} + M_s\Big(\sum_{k+1}^{j} X_{t_i} I_{E_i}\Big) + M_s\big(X_t I_{\{\tau \geq t_{j+1}\}}\big)$$

Recall that $X_{t_i} = M_{t_i}(X_\infty)$ and since $E_i \in \mathcal{F}_{t_i}$ then, $X_{t_i} I_{E_i} = M_{t_i}(X_\infty E_{t_i})$

so,

$$M_s\Big(\sum_{k+1}^{j} X_{t_i} I_{E_i}\Big) = \sum_{k+1}^{j} M_s(M_{t_i}(X_\infty E_{t_i})) = M_s\Big(X_\infty \sum_{k+1}^{j} I_{E_i}\Big)$$

while $M_s(X_t I_{\{\tau \geq t_{j+1}\}}) = M_s(M_t(X_\infty) I_{\{\tau \geq t_{j+1}\}})$

$= M_s\left(M_t\left(X_\infty I_{\{\tau \geq t_{j+1}\}}\right)\right) = M_s\left(X_\infty I_{\{\tau \geq t_{j+1}\}}\right)$

since $\{\tau \geq t_{j+1}\} = \Omega \setminus \{\tau \leq t_j\} \in \mathcal{F}_t$. So

$$M_s(X_{\tau \wedge t}) = \sum_{i=0}^{k} X_{t_i} I_{E_i} + M_s\left(X_\infty I_{\{\tau \geq t_{k+1}\}}\right)$$

$$= \sum_{i=0}^{k} X_{t_i} I_{E_i} + M_s(X_\infty) I_{\{\tau \geq t_{k+1}\}}$$

as $\{\tau \geq t_{k+1}\} = \Omega \setminus \{\tau \leqslant t_k\} \in \mathcal{F}_s$.

$$= \sum_{i=0}^{k} X_{t_i} I_{E_i} + X_s I_{\{\tau \geq t_{k+1}\}} = X_{\tau \wedge s}.$$

So $(X_{\tau \wedge t})$ is a martingale. This works for all τ.

CB Do an approximation proof. Note also $(X_{\tau \wedge t}) = (M_t(X_\tau))$

1.3 **Lemma** Suppose that $X, Y \in \mathcal{M}_0^2$ and for every stopping time of the filtration, typically τ, we have X_τ is orthogonal to Y_τ in $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$. Then $X \perp Y$.

__Proof__

We have $\mathbb{E}(X_\tau Y_\tau) = \mathbb{E}((XY)_\tau) = 0$ for all stopping times. Let $t \in [0, \infty)$ and $E \in \mathcal{F}_t$. Let $\tau = t I_E + \infty I_{\Omega \setminus E}$. Thus is a stopping time of the filtration:

$$\{\tau \leqslant s\} = \begin{cases} \emptyset & \text{if } s < t \text{ and } \emptyset \in \mathcal{F}_s \\ E & \text{if } t \leqslant s < T \text{ and } E \in \mathcal{F}_s \\ \Omega & \text{if } t = \infty \text{ and } \Omega \in \mathcal{F}_\infty \end{cases}$$

Now $(XY)_\tau = X_t Y_t I_E + X_\infty Y_\infty I_{\Omega \setminus E}$

and $\mathbb{E}((XY)_\tau) = \mathbb{E}(X_t Y_t I_E) + \mathbb{E}(X_\infty Y_\infty I_{\Omega\setminus E})$.

We also have,

$$\mathbb{E}(X_\infty Y_\infty) = \mathbb{E}(X_\infty Y_\infty I_E) + \mathbb{E}(X_\infty Y_\infty I_{\Omega\setminus E})$$

Now, by hypothesis,

$0 = \mathbb{E}(X_\infty Y_\infty) = \mathbb{E}((XY)_\tau)$, which leads to

$$\mathbb{E}(X_t Y_t I_E) = \mathbb{E}(X_\infty Y_\infty I_E) \quad \forall E \in \mathcal{F}_t.$$

Recalling the Radon Nickodym theorem this amounts to the statement that

$$M_t(X_\infty Y_\infty) = X_t Y_t, \quad 0 \leq t \leq T.$$

i.e. $XY \in \mathcal{M}^1$.

The converse is also true, the proof I have uses optional stopping. See later!

Using the correspondence between $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and $\mathcal{M}^2$ we can define a norm and inner product on $\mathcal{M}^2$ — we just pull them back from $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ onto $\mathcal{M}^2$:

$$L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \ni U \longmapsto (M_t(U)) \in \mathcal{M}^2$$

so $\|(M_t(U))\| = \|U\|_2$ and

$$\langle (M_t(U)), (M_t(V)) \rangle = \langle U, V \rangle.$$

Clearly, $\mathcal{M}^2$ is a vector space and, with the definitions above, it is a Hilbert space.

Here is a key idea:

1.4 Definition A closed subspace, F, of $\mathcal{M}^2$, is called <u>stable</u> if it is stable under stopping, that is,

If $(X_t) \in F$ and τ is a stopping time then $X^\tau \equiv (X_{\tau \wedge t}) \in F$.

Note that since $\mathcal{F}_0 = \overline{\{\Omega, \phi\}}$ then for each $E \in \mathcal{F}_0$ and $(x_t) \in F$, $(X_t I_E)$ is also in F. In a more general setting this property would be part of the definition.

* Stopping a martingale gives a martingale.

There is a deep relationship between stopping and stochastic integration. One can take stopping as a departure (+) point for developing stochastic integration. Because of this a subspace of $\mathcal{M}^2$ which is stable under stopping is also stable under stochastic integration for suitable integrands. In fact there are a number of characterisations of stable subspaces:

1.5 Lemma.

The following are equivalent for a closed subspace, F, of $\mathcal{M}^2$.

* Borrowed from Philip Protters book.

(+) Random Times. Predictable.... Proc, London Math Soc (3) 67 (1993)

(i) F is a (closed) stable subspace of $\mathcal{M}^2_0$.

(ii) For $t \in [0,T]$ and $E \in \mathcal{F}_t$ and $X \in F$ the process $(X - X^t)I_E$ belongs to F too.

(iii) If $X \in F$ and (h_s) is a bounded predictable process then

$$\left(\int_0^t h_s dX_s \right)$$

belongs to F too.

(iv) If $X \in F$ and (h_s) is a predictable process such that,

$$\mathbb{E} \left(\int_0^T h_s^2 \, d\langle X, X \rangle_s \right) < \infty$$

then

$$\int_0^T h_s dX_s$$

belongs to F too.

Proof Let $\tau = tI_E + \infty I_{\Omega \setminus E}$ where $E \in \mathcal{F}_t$.

For $X \in F$, $X^\tau \equiv (X_{\tau \wedge s})$ where

$$X_{\tau \wedge s} = \begin{cases} X_s & \text{if } s < t, \\ X_t I_E + X_s I_{\Omega \setminus E}, & \text{for } t \le s \le \infty \end{cases}$$

So $$X_s - X_{\tau \wedge s} = \begin{cases} 0 & \text{if } s < t \\ (X_s - X_t) I_E & \text{for } t \le s \le \infty \end{cases}$$

and this is $(X - X^\tau) I_E$, so it's in F. (i)$\Rightarrow$(ii)

A moments thought shows that $(X-X^{\tau})I_E$ is simply the process obtained from a stochastic integral,

$$(X-X^{\tau})I_E = \left(\int_0^s f(r)\,dX_r \right)$$

where f is the elementary process

$$f(s) = I_E \, I_{[t,\infty)}(s) \, .$$

Because F is a subspace, linear combinations of such f's can be assembled and we conclude

$$\left(\int_0^s f(r)\,dX_r \right) \in F$$

whenever $f(s) = h\, I_{[t,\infty)}(s)$ and h is a simple $\mathcal{F}_t$ random variable. Now because F is closed the stochastic integrals of (appropriate) limit of these processes will be in F, that is,

$$\left(\int_0^s f(r)\,dX_r \right) \in F$$

whenever $f(s) = k\, I_{[t,\infty)}(s)$ and k is a bounded $\mathcal{F}_t$ random variable.

Now we let 't' vary and using linearity and the fact that F is a subspace we conclude that

$$\left(\int_0^s f(r)\,dX_r \right) \in F$$

whenever f is a simple predictable process.
Now the integral for a bounded predictable process is a limit, at each time t, of the integrals of a sequence of simple predictable processes, and the convergence is in L^2 norm. As a consequence

$$\left(\int_0^s f(r)\,dX_r\right) \in F$$

For every bounded predictable process, f. So (ii) $\Rightarrow$ (iii). A similar argument shows that (iii) $\Rightarrow$ (iv).

Now consider a time, τ, taking just two values, $0 \leq u < v \leq \infty$, on sets E and $\Omega \setminus E$. For any martingale $X \in F$,

$$X^\tau \equiv (X_{\tau \wedge s}) = \begin{cases} X_s & \text{if } s \leq u \\ X_u I_E + X_s I_{\Omega\setminus E}, & u < s \leq v \\ X_u I_E + X_v I_{\Omega\setminus E} & v < s \end{cases}$$

So let f be the simple predictable process

$$f(s) = \begin{cases} I_\Omega & \text{if } s \leq u \\ I_\Omega - I_E & \text{if } u < s \leq v \\ 0 & \text{if } v < s \end{cases}$$

then, as $X \in \mathcal{M}_0^2$,

$$\int_0^s f(r)\,dX_r = \begin{cases} X_s - X_0 = X_s & \text{if } s \le u \\ X_u + (X_s - X_u) - (X_s - X_u) I_E & \text{if } u < s \le v \\ X_u + (X_v - X_u) - (X_v - X_u) I_E & \text{if } v < s \end{cases}$$

$$= \begin{cases} X_s & \text{if } s \le u \\ X_s - (X_s - X_u) I_E = X_u I_E + X_s I_{\Omega \setminus E} & u < s \le v \\ X_u I_E + X_v I_{\Omega \setminus E} & \text{if } v < s \end{cases}$$

$= X^\tau$. So (vi) $\Rightarrow$ (i) in this case.

One can extend this result to times, τ, which take a finite number of distinct values. Moreover, any time τ is the limit of a decreasing sequence of such times. (Exercise). One can prove that for $X \in \mathcal{M}_0^2$,

$$X_{\tau_n} \to X_\tau \quad \text{in } L^2 \text{ norm.}$$

All of which shows, (iv) $\Rightarrow$ (i).

Let $\mathcal{a} \subseteq \mathcal{M}_0^2$. Then as $\mathcal{M}_0^2$ is stable under stopping and a closed subspace it follows that $\langle \mathcal{a} \rangle = \bigcap \{F : F \text{ is closed \& stable}, F \supseteq \mathcal{a}\}$ exists. We call it the stable subspace generated by $\mathcal{a}$. Let $\mathcal{a} = \{X\}$ where $X \in \mathcal{M}_0^2$. Consider $\langle X \rangle \equiv \langle \{X\} \rangle$. From the result above, part (iv), we know that for every predictable f such that $\mathbb{E}(\int_0^\infty f_s^2 \, d\langle X \rangle_s) < \infty$ the martingale,

$$(Z_t) = \left(\int_0^t f(s)\, dX_s\right)$$

lies in $\langle X \rangle$. But as we saw in the proof of (iv) $\Rightarrow$ (i), stopping can be expressed as stochastic integration. So stopping the martingale (Z_t) amounts to integrating $I_\Omega - I_{\{\tau \leq t\}}$ with respect to (Z_t), that is

$$Z_{\tau \wedge t} = \int_0^{\tau \wedge t} (I_\Omega - I_{\{\tau \leq s\}})\, dZ_s = \int_0^{\tau \wedge t} (I_\Omega - I_{\{\tau \leq s\}}) f(s)\, dX_s$$

So, if you stop a martingale obtained by stochastic integration, it is itself a stochastic integral. This says that these stochastic integrals are stable under stopping! Therefore

$$\langle X \rangle = \left\{ \left(\int_0^t h\, dX\right) : \mathbb{E}\left(\int_0^\infty h_s^2\, d\langle X \rangle_s\right) < \infty \right\}.$$

Let F be a stable subspace of $\mathcal{M}^2_0$. Write $F^\perp = \{N : N \perp K, \forall K \in F\}$. So $F^\perp$ is the strong orthogonal of F. We are going to look at the case where F is $\langle W \rangle$, W Brownian Motion, and $(\Omega, \mathcal{F}_\infty, \mathbb{P}, (\mathcal{F}_t), [0,\infty))$ is the (completed) filtration given by Brownian Motion. Since W is not in $\mathcal{M}^2_0$, we need to localise "everything" to $[0,T]$ for $T < \infty$. So our filtration is indexed by $[0,T]$, the definitions and results above carry over to this "smaller" filtration with appropriate modifications. We will write $\mathcal{M}^2_0(T)$ to indicate the restriction to $[0,T]$. Before we proceed, a theorem about $\langle W \rangle^\perp$.

Theorem

Let F be a stable subspace of $\mathcal{M}_0^2$ and $F^\perp$ its strong orthogonal. Then $F^\perp$ is a closed stable subspace of $\mathcal{M}_0^2$.

Pf Let $X \in F$ and $M, N \in F^\perp$ and $\alpha, \beta \in \mathbb{R}$.

(i) $X(\alpha M + \beta N) = \alpha XM + \beta XN$ a sum of martin-gales so $\alpha M + \beta N \in F^\perp$.

(ii) If $(X^n) \subset F^\perp$ and $X^n \to X$ in $\mathcal{M}_0^2$ then $X^n_\infty \to X_\infty$ in $\|\cdot\|_2$. So $X^n_t \to X_t$ in $\|\cdot\|_2$ for each t. Let $N \in F$. Then,

$$X^n_t N_t \to X_t N_t \quad \text{in } L^1 \text{ norm for } t \in [0, \infty]$$

since $(X^n_t N_t)$ is an L^1-martingale $(X_t N_t)$ is an L^1-mart, so $X \in F^\perp$. So $F^\perp$ is a closed subspace.

(iii) Let τ be a stopping time taking the distinct values $t_0, t_1, \ldots, t_n$. We have already seen that for a martingale $N \in F^\perp$

$$N_{\tau \wedge t} = \sum_{i=0}^{j} N_{t_i} I_{E_i} + N_t I_{\{\tau \geq t_{j+1}\}}$$

where $t_j \leq t < t_{j+1}$ and $E_i = \{\tau = t_i\} \in \mathcal{F}_{t_i}$,

Let $X \in F$. Then XN is an L^1 martingale and it is exactly $(M_t(X_\infty N_\infty))$. Let $t_k \leq s < t_{k+1}$, $s < t$,

$$M_s(X_t N_{\tau \wedge t}) = M_s(X_t \sum_0^k N_{t_i} I_{E_i}) + M_s(X_t \sum_{k+1}^j N_{t_i} I_{E_i}) + M_s(X_t N_t I_{\{\tau \geq t_{j+1}\}})$$

Now, $M_s(X_t \sum_{k+1}^j N_{t_i} I_{E_i}) = \sum_{k+1}^j M_s(X_t N_{t_i} I_{E_i})$

$$= \sum_{k+1}^j M_s(M_{t_i}(X_t N_{t_i} I_{E_i})) = \sum_{k+1}^j M_s(M_{t_i}(X_t) N_{t_i} I_{E_i})$$

$$= \sum_{k+1}^j M_s(X_{t_i} N_{t_i} I_{E_i}) = \sum_{k+1}^j M_s(M_{t_i}(X_\infty N_\infty) I_{E_i})$$

$$= \sum_{k+1}^j M_s(M_{t_i}(X_\infty N_\infty I_{E_i})) = \sum_{k+1}^j M_s(X_\infty N_\infty I_{E_i}).$$

Also

$$M_s(X_t N_t I_{\{\tau \geq t_{j+1}\}}) = M_s(M_t(X_\infty N_\infty) I_{\{\tau \geq t_{j+1}\}})$$

since $\{\tau \geq t_{j+1}\} = \Omega \setminus \{\tau \leq t_j\} \in \mathcal{F}_{t_j}$ and $t \geq t_j$

$$= M_s(M_t(X_\infty N_\infty I_{\{\tau \geq t_{j+1}\}})) = M_s(X_\infty N_\infty I_{\{\tau \geq t_{j+1}\}})$$

So, $M_s\left(X_t \sum_{k+1}^{j} N_{t_i} I_{E_i}\right) + M_s\left(X_t N_t I_{\{\tau \geq t_{j+1}\}}\right)$.

$= M_s\left(X_\infty N_\infty \left(\sum_{k+1}^{j} I_{E_i} + I_{\{\tau \geq t_{j+1}\}}\right)\right)$

$= M_s\left(X_\infty N_\infty I_{\{\tau \geq t_{k+1}\}}\right)$ but (again)

$\{\tau \geq t_{k+1}\} \in \mathcal{F}_{t_k}$ and $t_k \leq s$, therefore

$= M_s(X_\infty N_\infty) I_{\{\tau \geq t_{k+1}\}} = X_s N_s I_{\{\tau \geq t_{k+1}\}}$

while,

$$M_s\left(X_t \sum_{i=0}^{k} N_{t_i} I_{E_i}\right) = M_s(X_t) \sum_0^k N_{t_i} I_{E_i} = X_s \sum_0^k N_{t_i} I_{E_i}.$$

So, $M_s(X_t N_{\tau\wedge t}) = X_s N_{\tau \wedge s}$, i.e. $F^\perp$ is stable under stopping by times which take a finite number of values. Once again, any τ is the limit of a decreasing sequence of times, τ_n, taking only finitely many values, and

$$N_{\tau_n} \longrightarrow N_\tau \quad \text{in } L^2$$

(and for each t, $N_{\tau_n \wedge t} \longrightarrow N_{\tau\wedge t}$ in L^2) this shows that $(N_{\tau\wedge t})$ is the limit of a sequence in $F^\perp$, which, since $F^\perp$ is closed, means $N_\tau \in F^\perp$, i.e. $F^\perp$ is stable.